GENERALIZED MARKUSHEVICH BASES

BY

J. A. DYER

ABSTRACT

Conditions for the embedding of a Banach space as a dense subspace of a continuous function space are studied. Necessary and sufficient conditions for a Banach space to be isomorphic to a P_1 space are found.

1. **Introduction.** Suppose V is a Banach space. A biorthogonal collection $\{(b_i,f_i)\}_{i\in I}$ is said to be a Markushevich basis for V provided $B=\{b_i\}_{i\in I}$ is fundamental in V and $\gamma=\{f_i\}_{i\in I}$ is total over V. The purpose of this paper is to consider an extension of this idea which results from assuming that the collection (B,γ) under study satisfies only an extended biorthogonality condition introduced by Kaplan in [4].

In this paper attention is restricted to the problem of embedding a Banach space which admits a generalized Markushevich basis as a dense subspace of a continuous function space (Theorem 1). One obtains as a corollary to this embedding theorem the result that there exist Banach spaces which do not admit Markushevich bases. In particular if Γ is an uncountable set $m(\Gamma)$ admits no Markushevich basis. This has apparently not been previously noted in the literature. Conditions for the mapping of Theorem 1 to be an onto map are found (Theorem 2). Application of Theorem 2 is made to some problems in measure theory (Example 1).

A Banach space V is said to be a P_λ space if whenever V is contained in a Banach space U there exists a projection of norm less than or equal to λ of U onto V. As a final application of the embedding theorems a set of necessary and sufficient conditions for a Banach space to be isomorphic to a P_1 space is obtained (Theorem 3).

2. **Notations and basic definitions.** In this paper only Banach spaces will be considered although many of the basic concepts have meaning in a more general setting. The Banach spaces may be over either the real or the complex field,

Received January 4, 1969, and in revised form March 17, 1969.

and the term scalar will be used to refer to either of these fields. If V is a linear space the zero functional on V will be denoted by θ. If V is a linear space and S is a subset of V then sp S will denote the span of S. All topological spaces considered here are assumed to be Hausdorff spaces. If H is a locally compact space then $C_0[H]$ will denote the uniform closure of the scalar valued continuous functions of compact support on H. Most of the terminology of this paper will be that of [3]. For convenience, some of the definitions from [3] and [4] are reproduced here.

Definition 1. ([3], Definition 1.) Suppose X is a non-void set and P is a collection of subsets of X. The statement that P is a proto-ring means that if each of A and B is in P then there exist finite disjoint collections, $\{E_i\}_{i=1}^{p}$ and $\{F_j\}_{j=1}^{q}$, of elements of P such that $A \cap B = \bigcup_{i=1}^{p} E_i$ and $A \sim B = \bigcup_{j=1}^{q} F_j$. If P contains a finite disjoint collection $\{G_k\}_{k=1}^{r}$ such that $X = \bigcup_{k=1}^{r} G_k$ then P will be said to be a proto-algebra.

Definition 2. ([4], Section 2.) Suppose V is a Banach space and B and γ are subsets of V and V^* respectively. The pair (B,γ) will be said to be biorthogonal in the wide sense (abbreviated as bows) if it satisfies the following conditions I and II:

I. a) If $b \in B$ and $f \in \gamma$ then $f(b)$ is zero or one;

b) No f is zero at every b, and for each b in B there is some f in γ such that $f(b)$ is one.

A finite subset $\{b_j\}_{j=1}^{k}$ of B will be said to be an γ-orthogonal set if no f in γ has value one at more than one b_j, $j = 1, 2, \cdots, k$.

II. The γ-orthogonal sets of B form a directed set under the ordering $\prec$ defined by:

$$\{b_j\}_{j=1}^{k} \prec \{b_i'\}_{i=1}^{q} \text{ iff } \{b_j\}_{j=1}^{k} \subset \mathrm{sp}\{b_i'\}_{i=1}^{q}.$$

Definition 3. ([3], Definition 5.). Suppose V is a Banach space and $(B \subset V, \gamma \subset V^*)$ is bows. An γ-orthogonal subset $\{b_j\}_{j=1}^{k}$ of B will be said to be a full γ-orthogonal subset if every element of γ has value one at some b_j, $j = 1, 2, \cdots, k$.

Given Kaplan's extension of the concept of biorthogonality one has an obvious extension of the concept of a Markushevich basis.

Definition 4. Suppose V is a Banach space and $(B \subset V, \gamma \subset V^*)$ is bows. The statement that (B,γ) is a generalized Markushevich basis for V means that B is fundamental in V and γ is total over V.

It might be noted that the definitions of generalized and dual generalized bases can be extended in the same way as has been done here for Markushevich bases. The full theory of similar bases as developed by Davis and Arsove can be very simply extended to the resulting structures. There seems however to be little point to such extensions and they will not be further considered here.

For the remainder of this paper it will be assumed that V is an infinite dimensional Banach space, B is a subset of V, γ is a subset of V^*, and $\bar{\gamma}$ is the weak star closure of γ.

3. **The basic embedding theorem.** The primary purpose ot this section is to present a proof for the following theorem:

THEOREM 1. *Suppose V is a Banach space and (B,γ) is a generalized Markushevich basis for V. If γ is norm bounded then there exists a Boolean space H and a continuous one-to-one map ε of V into $C_0[H]$ such that $\varepsilon[V]$ is dense in $C_0[H]$. Furthermore H is compacy if and only if B admits a full γ-orthogonal set.*

In order to facilitate the proof of Theorem 1 it is helpful first to consider two preliminary lemmas.

LEMMA 1. *If (B,γ) is bows and B is fundamental then (B,γ') is also bows, where γ' denotes $\bar{\gamma} \sim \{\theta\}$. Furthermore a finite subset of B is (full) γ-orthogonal if and only if it is (full) γ'-orthogonal. If B admits a full γ-orthogonal set then θ is not a w^*-limit point of γ.*

Proof. Since B is fundamental, it follows from the definition of γ' that (B,γ') satisfies Condition I of Definition 2. If $\{b_i\}_{i=1}^k$ is an γ'-orthogonal subset of B then it is obviously γ-orthogonal. If $\{b_i\}_{i=1}^k$ is γ-orthogonal it follows that no element f of $\gamma' \sim \gamma$ can have the value 1 at two elements of $\{b_i\}_{i=1}^k$ since each such f is the pointwise limit of a net in γ. Therefore a finite subset of B is γ-orthogonal if and only if it is γ'-orthogonal. A similar argument yields the result that a finite subset of B is full γ-orthogonal if and only if it is full γ'-orthogonal, and that if B admits a full γ-orthogonal subset then θ is not a w^*-limit point of γ. This completes the proof.

It might be noted in passing that if (B,γ) is biorthogonal rather than just bows then γ' is γ, since θ is the only w^*-limit point of γ in this case.

LEMMA 2. *Suppose (B,γ) satisfies the conditions of Lemma 1 and that γ is norm-bounded. Then γ' with the relativized w^*-topology is a Boolean space.*

Furthermore γ' is w^*-compact if and only if B admits a full γ-orthogonal set.

Proof. For each b in B, let $E(b)$ be the set $\{f : f \in \gamma' \text{ and } f(b) = 1\}$. It follows from Lemma 1 and the proof of Theorem 4 in [3] that if B admits a full γ-orthogonal subset then $P = \{E(b) : b \in B\} \cup \{\Phi\}$ is a proto-algebra of subsets of γ. If B does not admit a full γ-orthogonal set then it may be shown by a straightforward argument based upon Proposition 3 of [4] that P is a proto-ring. It is also an immediate consequence of Definitions 1 and 3 that if P is a proto-algebra then B admits a full γ-orthogonal subset.

Since P is a proto-ring, finite intersection of elements in P can be written as finite disjoint unions of elements in P and therefore P is the base for a topology, τ, for γ'. It is clear from the definition of P that τ is weaker than the relativized w^*-topology on γ'. Since (B, γ') is bows by Lemma 1, it follows from Condition II of Definition 2, that τ is Hausdorff. Since γ is norm bounded it follows that any w^*-closed subset of γ' is w^*-compact. Suppose $\bar{b}$ is an element of B and consider the w^*-closure of $E(\bar{b})$. Since $f(\bar{b}) = 1$, $\forall f \in E(\bar{b})$, θ is not a w^*-limit point of $E(\bar{b})$ and it follows that $E(\bar{b})$ is w^*-closed. Hence $E(\bar{b})$ is w^*-compact and so the w^* and τ-topologies agree on $E(\bar{b})$. From this it follows that the w^* and τ-topologies agree on γ', and that P is a base of compact open sets for the relativized w^*-topology on γ'. Hence γ' with the relativized w^*-topology is a Boolean space. If θ is not a w^*-limit point of γ then γ' is w^*-closed and hence compact. In this case it follows that some finite disjoint collection of elements in P covers γ', or in other words P is a proto-algebra. This completes the proof.

There is another way in which one may characterize γ' with the relativized w^*-topology. It follows from Theorem 1.1 of [7] that the ring generated by P, $R(P)$, is the collection of all finite disjoint unions of elements of P. Hence every set in $R(P)$ is a compact open subset of γ' with the relativized w^*-topology. Conversely since P is a base of compact open sets for this topology, every compact open set is in $R(P)$. It therefore follows that γ' with the relativized w^*-topology is homeomorphic to the Stone space of $R(P)$.

If (B, γ) is a biorthogonal collection which satisfies the conditions of Lemma 2 it is an immediate consequence of the proof of Lemma 2 and the observation that γ' is γ that the relativized w^*-topology on γ is the discrete topology.

Proof of Theorem 1. Let H denote γ' with the relativized w^*-topology and let ε be the mapping of V into $C_0[H]$ defined by:

$$\varepsilon v(f) = f(v), \qquad \forall f \in \gamma'.$$

Since γ is total and norm bounded it follows that ε is one-to-one and continuous. If $b \in B$ then εb is $\chi_{E(b)}$. Since P is a proto-ring it follows that if b and b' are in B then there exists an element v in sp B such that $\varepsilon v = \varepsilon b \cdot \varepsilon b'$. Since θ does not belong to γ' and B is fundamental in V, $\varepsilon[B]$ does not vanish identically at any point of γ'. The hypothesis that B is fundamental also implies that $\varepsilon[B]$ separates the points of γ'. Since $\varepsilon[B]$ contains only real valued functions it follows from the Stone-Weierstrass theorem that $\varepsilon[\operatorname{sp} B]$ is dense in $C_0[H]$. The remaining assertions of the theorem are consequences of Lemma 2.

If V is a Banach space which admits a Markushevich basis then it is easy to see that V admits a Markushevich basis (B,γ) which satisfies the conditions of Theorem 1; the comments following Lemma 2 then imply that V may be continously embedded as a dense subset of a c_0 space, namely $C_0[\gamma]$. It then follows from a comment of Day in [2] (p. 518, (3)) that every Banach space with a Markushevich basis is isomorphic to a strictly convex space. In view of the results of [2] this gives one way to show that there exist Banach spaces which do not admit Markushevich bases. In particular if Γ is uncountable $m(\Gamma)$ admits no Markushevich basis ([2], Theorem 8, Corollary).

4. Some applications of Theorem 1.

THEOREM 2. *Suppose that (B,γ) satisfies the conditions of Theorem 1. In order that the mapping ε of Theorem 1 be an isomorphism it is necessary and sufficient that there exist a positive number λ with the property that for each γ-orthogonal set $\{b_i\}_{i=1}^p$ and scalar sequence $\{\alpha_i\}_{i=1}^p$,*

$$\left\| \sum_{i=1}^{p} \alpha_i b_i \right\| \leqq \lambda \sup_{1 \leqq i \leqq p} |\alpha_i|.$$

Proof. If $\{b_i\}_{i=1}^p$ is an γ-orthogonal set then $E(b_i) \cap E(b_j)$ is empty when b_i is not b_j. It follows therefore that $\operatorname{sp}\{\varepsilon b_i\}_{i=1}^p$ is isometrically isomorphic to l_p^∞. Let $\{C_\alpha\}_{\alpha \in A}$ denote the collection of γ-orthogonal subsets of B, and for each α, let D_α denote $\operatorname{sp} C_\alpha$. It follows from the definition of γ-orthogonality that the collection $\{D_\alpha\}_{\alpha \in A}$ is directed by set inclusion and that $\bigcup_{\alpha \in A} D_\alpha$ is dense in V. It follows from the proof of Theorem 1 that these same properties hold for $\{\varepsilon[D_\alpha]\}_{\alpha \in A}$. A Banach space V with the property that there exists a directed by inclusion family of subspaces $\{E_\alpha\}_{\alpha \in A}$ of V whose union is dense in V and such that each E_α is isometrically isomorphic to some $l_{n(\alpha)}^\infty$ is said by Michael and Pełczyński ([5])

to be a Π_1^∞-space. Thus $C_0[H]$ is a Π_1^∞-space. If ε is an isomorphism then it follows that there exists a positive number λ such that if $\{b_i\}_{i=1}^p$ is γ-orthogonal and $\{\alpha_i\}_{i=1}^p$ is a scalar sequence then

$$\left\| \sum_{i=1}^{p} \alpha_i b_i \right\| \leqq \lambda \sup_{1 \leqq i \leqq p} |\alpha_i|.$$

This condition is also sufficient, since if it is satisfied it is clear that $\varepsilon^{-1}/\bigcup_{\alpha\in A}\varepsilon[D_\alpha]$ is continuous and both $\bigcup_{\alpha\in A}D_\alpha$ and $\bigcup_{\alpha\in A}\varepsilon[D_\alpha]$ are dense subsets of their respective spaces. This completes the proof.

It might be noted that since $C_0[H]$ is a Π_1^∞-space it follows from a comment in [5], p. 189, that if V is reflexive then ε cannot be an isomorphism.

A bows collection which satisfies the final hypothesis of Theorem 2 will be said to have property III. If B admits a full γ-orthogonal subset then Theorem 2 holds if "γ-orthogonal" is replaced by "full γ-orthogonal" in the statement of the theorem. This follows from the fact that any γ-orthogonal set is followed by a full γ-orthogonal set in the ordering $\prec$ (cf. [3], proof of Theorem 4). It follows therefore that the subspaces of $\{D_\alpha\}_{\alpha\in A}$ which correspond to full γ-orthogonal sets are cofinal in the complete collection.

EXAMPLE 1. Suppose X is a non-void set and P a proto-ring of subsets of X. Let $Q(X,P)$ denote the collection of scalar valued functions on X which are uniformly approximatible by finite linear combinations of characteristic functions of sets in P. This usage of the symbol $Q(X,P)$ differs from the usage introduced in [7]. It is clear that $Q(X,P)$ with the sup norm topology is a Banach space. Let B denote the collection of characteristic functions of non-void sets in P, and let γ denote the collection of evaluation functionals; i.e. the collection $\{\phi_x : x \in X\}$, where ϕ_x is defined by $\phi_x(g) = g(x)$, $\forall g \in Q(X,P)$. It follows by similar arguments to those in [3], Section 4, that (B,γ) is bows and hence a generalized Markushevich basis for $Q(X,P)$, that B admits a full γ-orthogonal set if and only if P is a proto-algebra, and that if $\{b_i\}_{i=1}^p$ is an γ-orthogonal subset of B then $\mathrm{sp}\{b_i\}_{i=1}^p$ is isometrically isomorphic to l_p^∞. Hence there exists a Boolean space H, H being compact if P is a proto-algebra, such that $Q(X,P)$ is isomorphic to $C_0[H]$. It is not difficult to show that in this case ε is actually an isometry.

It has been shown by other methods that if P is an algebra ([6]) or a proto-algebra ([7]) then $Q(X,P)$ is isometrically isomorphic to $C[H]$, for some compact Hausdorff space H. The result of Example 1 in the case that P is a ring or proto-ring has apparently not been previously noted in the literature.

Other related results from measure theory may be obtained by these same methods. For example, Theorem 4.3 of Yosida and Hewitt in [8] which asserts that $L_\infty(T,\mathscr{M},\mathscr{N})$ ([8], Definition 2.1) is isometrically isomorphic to $C[H]$, for some compact space H, may be obtained from Theorem 2 by taking for B the equivalence classes generated by characteristic functions of sets in $\mathscr{M} \sim \mathscr{N}$ and for γ the functionals generated by the ω measures of Theorem 4.1, [8].

Since it is known that every real P_1 space is isometrically isomorphic to $C[H]$ where H is compact and extremally disconnected; cf. [1], Theorem 3, p. 94; and since a compact extremally disconnected space is Boolean, it is of some interest to consider conditions under which the space H of Theorem 1 is in fact extremally disconnected, assuming it is compact. It follows from the remarks following Lemma 2 that a subset S of H is open and closed if and only if there exists an γ-orthogonal set $\{b_i\}_{i=1}^p$ such that S is $\bigcup_{i=1}^p E(b_i)$. Hence the condition that H be extremally disconnected is equivalent to the condition that for each subset B' of B there exists an γ-orthogonal set $\{b_i\}_{i=1}^p$ such that

$$\overline{\bigcup_{b \in B'} E(b)} = \bigcup_{i=1}^{p} E(b_i),$$

closures being taken in the relativized w^*-topology. This in turn is clearly equivalent to the condition that an element f of γ' annihilates $\bigcup_{i=1}^p E(b_i)$ if and only if every net in γ' which w^*-converges to f contains a subnet each element of which annihilates B'. A bows collection (B,γ) which satisfies this condition will be said to have property IV.

These observations suggest:

THEOREM 3. *Suppose that V is a real Banach space. A necessary and sufficient condition that V be isomorphic to a P_1 space is that there exist a generalized Markushevich basis (B,γ) for V such that:*

1) *γ is w^*-compact,*
2) *(B,γ) has properties* III *and* IV.

Proof. The sufficiency of the desired conditions is an immediate consequence of Theorem 2 and the discussion preceeding the definition of property IV.

Now suppose H is a compact extremally disconnected space. Let B be the subset of $C[H]$ to which a function g belongs if and only if g is the characteristic function of a compact open subset of H, and let γ denote the evaluation functionals on $C[H]$. Since (B,γ) is clearly a generalized Markushevich basis for $C[H]$ and since B admits a full γ-orthogonal set it follows from Lemmas 1 and 2 that (B,γ') is a generalized Markushevich basis for $C[H]$ with γ' w^*-compact. It can be shown; cf. the remarks following Example 1, [3]; that $C[H]$ is $Q(H,P)$ where P is the algebra of compact open subsets of H. It therefore follows that $C[H]$ is isometrically isomorphic to $C_0[\gamma']$ and so H and γ' are homeomorphic. It then follows that (B,γ') has property IV. Therefore, if V is isomorphic to $C[H]$, $(\mathscr{I}^{-1}[B], \mathscr{I}^*[\gamma'])$, where $\mathscr{I}$ denotes an isomorphism on V onto $C[H]$, is a generalized Markushevich basis for V such that $\mathscr{I}^{-1}[B]$ admits a full $\mathscr{I}^*[\gamma']$-orthogonal set, $\mathscr{I}^*[\gamma']$ is w^*-compact, and $(\mathscr{I}^{-1}[B], \mathscr{I}^*[\gamma'])$ has property IV. Since the γ-orthogonal subsets of B determine a Π_1^∞ decomposition for $C[H]$ it follows that $(\mathscr{I}^{-1}[B], \mathscr{I}^*[\gamma'])$ has property III. This completes the proof.

5. **Generalizations.** An examination of the proofs of the lemmas and theorems of the previous sections show that the full force of the assumption that (B,γ) is bows is used only to show that γ' is Boolean in the relativized w^*-topology. This suggests the following generalization:

DEFINITION 5. Suppose V is a Banach space. The statement that $(B\subset V, \gamma\subset V^*)$ is pseudo-bows means that 0 does not belong to B and:

I. If $(b_1,b_2)\in B\times B$ then there exists v in sp B such that

$$f(v) = f(b_1)\cdot f(b_2),\ \forall f\in\gamma.$$

II. The finite linearly independent subsets of B are directed by $\prec$.

It is not difficult to show that if (B,γ) is pseudo-bows and B is fundamental then (B,γ') is also pseudo-bows. It can then be shown by methods almost identical to those used in the proof of Theorem 1 that if γ is total over V and norm bounded, and ε is defined as in Theorem 1, then ε is a one-to-one continuous map of V into a dense subspace of $C_0[\gamma']$, where γ' is assumed to be topologized with the relativized w^*-topology. Furthermore it follows in the same manner as in Theorem 2 that ε is onto, and hence an isomorphism, if and only if there exists a positive number λ such that if $\{b_i\}_{i=1}^p$ is a linearly independent subset of B and $\{\alpha_i\}_{i=1}^p$ is a scalar sequence then

$$\left\| \sum_{i=1}^p \alpha_i b_i \right\| \leqq \lambda \sup_{1\leqq i\leqq p} |\alpha_i|.$$

In conclusion, it seems only fair to state that the author has been unable to find any applications of this generalization.

References

1. M. M. Day, *Normed Linear Spaces*, Academic Press, New York, 1962.

2. M. M. Day, *Strict convexity and smoothness of normed spaces*, Trans. Amer. Math. Soc. **78** (1955), 516–528.

3. J. A. Dyer, *Integral bases in linear topological spaces*, to appear in Ill. J. Math.

4. S. Kaplan, *Biorthogonality and integration*, Proc. Amer. Math. Soc. **7** (1956), 109–114.

5. E. Michael and A. Pełczyński, *Separable Banach spaces which admit l_n^∞ approximations*, Israel J. Math. **4** (1966), 189–198.

6. P. Porcelli, *Adjoint spaces of abstract L_p spaces*, Port. Math. **25** (1966), 105–122.

7. W. L. Woodworth, *Integrals on proto-rings*, Iowa State University doctoral dissertation (1968).

8. K. Yosida and E. Hewitt, *Finitely additive measures*, Trans. Amer. Math. Soc. **72** (1952), 46–66.

Iowa State University of Science and Technology,
Ames, Iowa